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STRESS SINGULARITIES IN BONDED ANISOTROPIC MATERIALS†

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Abstract—In this paper, the general problem of stress singularities occurring near the free edge of two bonded anisotropic materials is investigated. After formulating the problem of two bonded anisotropic wedges, the stress singularity near the free edge of two bonded layers, at the tip of a crack between two materials and in the vicinity of a broken layer is obtained by simply varying the wedge angles. It is shown that, the power of singularity depends on the elastic properties of the anisotropic constituents. Several examples occurring in real layered composites structures and illustrating this fact are studied in some detail. Unlike the isotropic case the power of singularity at the tip of a crack between two materials can be real.

1. INTRODUCTION

In the recent past, research in anisotropic materials in general and in layered anistropic materials in particular has attracted a great deal of interest. The reasons for this are the ever-increasing use of composites in many structures and the fact that the failure resistance of composites can be improved considerably without adding to the weight of the structure. Indeed one attractive aspect of using composites is the possiblity of an economical design by reducing the weight of the structure. In layered composite structures, one of the most important design problems is the initiation and propagation of the so-called "delamination surface". From the study of bonded isotropic materials[1–3] it is well known that, the stresses near a free edge have a weak power singularity and under certain loading conditions, the normal stress can be tensile, thus enhancing the possibility of delamination failure. After the formation of new delamination surfaces, under sustained or excessive loading conditions the "crack" or "delamination surface" may further propagate in a stable or unstable manner. It is clear that, the severity of the power of singularity may have a decisive bearing upon the initiation and propagation of such delamination surfaces. Treated as single layers, the constituents of layered composites may in general be treated as anisotropic materials. Even though it is very difficult to determine all the material constants (a_{ij}) of an anisotropic material, fiber-composites are usually modeled as orthotropic materials, one of the principal axes of orthotropy coinciding with the direction of fibers. However, if the axes do not coincide with the fiber directions, the material becomes fully anisotropic in the new coordinate system and must be treated as such. The material constants are then found by a simple coordinate transformation. The existing solutions of bonded materials deal with either isotropic[1–3] or othotropic materials[4–7]. Lately, the problem of free edge singularities has been considered in[13]. In this paper, the stress state near the free edge of two bonded anisotropic materials will be studied by using Lekhnitskii's general formulation[8, 9] and Williams' method[10, 11]. First, the general asymptotic solution for two bonded wedges is given. Then by varying the wedge angles the stress singularities near the free-edge of two bonded anisotropic materials, near a crack tip at the interface of two materials and in the vicinity of a broken laminate are computed for different anisotropic materials obtained by varying the ply angle. Finally an example illustrating the dependence of the power of singularity on the wedge angle is given, and the application of the method to the finite element technique in formulating the special elements is discussed.

2. GENERAL FORMULATION

Consider two anisotropic wedges perfectly bonded along the x-axis (Fig. 1). In many applications the loads do not vary in z-direction and the dimension of the plate in the

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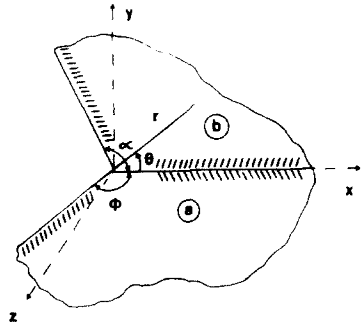

Fig. 1. Geometry of the problem.

z-direction is large enough to warrant the assumption that the stresses and displacements are independent of the z-coordinate. Since the materials are anisotropic the x–y plane is not a plane of symmetry. Therefore, despite the fact that the problem is two dimensional, the x–y plane will not remain plane after deformation, and the stress state will be three-dimensional. This type of deformation is referred to as "generalized plane strain" or "generalized plane deformation". For an anisotropic material, the stress–strain relations can be written as:

$$\begin{aligned} \epsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + \cdots + a_{16}\tau_{xy} \\ \epsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + \cdots + a_{26}\tau_{xy} \\ &\cdots\cdots\cdots\cdots \\ \gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + \cdots + a_{66}\tau_{xy} \end{aligned} \tag{2.1}$$

where $a_{ij}(i,\ j = 1,\ 6)$ are the elastic constants. Under the assumptions stated earlier we have $\epsilon_z = 0$ and the third equation of (2.1) gives:

$$\sigma_z = -\frac{1}{a_{33}}(a_{13}\sigma_x + a_{23}\sigma_y + a_{34}\tau_{yz} + a_{35}\tau_{xz} + a_{36}\tau_{xy}). \tag{2.2}$$

Substituting (2.2) into eqns (2.1), we obtain:

$$\begin{aligned} \epsilon_x &= \beta_{11}\sigma_x + \beta_{12}\sigma_y + \beta_{14}\tau_{yz} + \beta_{15}\tau_{xz} + \beta_{16}\tau_{xy} \\ \epsilon_y &= \beta_{12}\sigma_x + \beta_{22}\sigma_y + \beta_{24}\tau_{yz} + \beta_{25}\tau_{xz} + \beta_{26}\tau_{xy} \\ &\cdots\cdots\cdots\cdots \\ \gamma_{xy} &= \beta_{16}\sigma_x + \beta_{26}\sigma_y + \beta_{46}\tau_{yz} + \beta_{56}\tau_{xz} + \beta_{66}\tau_{xy} \end{aligned} \tag{2.3}$$

σ_z is given by (2.2) and, $\beta_{ij} = a_{ij} - a_{i3}\ a_{j3}/a_{33}$ $(i, j = 1, 2, 4, 5, 6)$. Since we are mainly concerned with the singular state in the vicinity of the wedge vertices, it is sufficient to consider the homogeneous part of the solution only. Thus, if the stresses are related to the stress functions $F(x,\ y)$ and $\psi(x,\ y)$ by means of the following expressions,

$$\sigma_x = \frac{\partial^2 F}{\partial y^2},\ \sigma_y = \frac{\partial^2 F}{\partial x^2},\ \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y},\ \tau_{xz} = \frac{\partial \psi}{\partial y},\ \tau_{yz} = -\frac{\partial \psi}{\partial x} \tag{2.4}$$

then the equilibrium equations are satisfied identically and the compatibility equations reduce to [8],

$$L_4 F + L_3\,\psi = 0 \tag{2.5}$$

$$L_3 F + L_2\,\psi = 0 \tag{2.6}$$

where the differential operators L_2, L_3, L_4 are defined by,

$$L_2 = \beta_{44}\frac{\partial^2}{\partial x^2} - 2\beta_{45}\frac{\partial^2}{\partial x\partial y} + \beta_{55}\frac{\partial^2}{\partial y^2} \tag{2.7}$$

$$L_3 = -\beta_{24}\frac{\partial^3}{\partial x^3} + (\beta_{25}+\beta_{46})\frac{\partial^3}{\partial x^2\partial y} - (\beta_{14}+\beta_{56})\frac{\partial^3}{\partial x\partial y^2} + \beta_{15}\frac{\partial^3}{\partial y^3} \tag{2.8}$$

$$L_4 = \beta_{22}\frac{\partial^4}{\partial x^4} - 2\beta_{26}\frac{\partial^4}{\partial x^3\partial y} + (2\beta_{12}+\beta_{66})\frac{\partial^4}{\partial x^2\partial y^2} - 2\beta_{16}\frac{\partial^4}{\partial x\partial y^3} + \beta_{11}\frac{\partial^4}{\partial y^4}. \tag{2.9}$$

Eliminating ψ from eqns (2.5) and (2.6) the following sixth order differential equation is obtained:

$$(L_4L_2 - L_3^2)F = 0. \tag{2.10}$$

The sixth-order operator can be decomposed into six linear operators of the first order, giving[8]:

$$D_6D_5D_4D_3D_2D_1F = 0 \tag{2.11}$$

where,

$$D_n = \frac{\partial}{\partial y} - \mu_n\frac{\partial}{\partial x} \tag{2.12}$$

the $\mu_n(n = 1, 2, \dots 6)$ being the roots of the following characteristic equation:

$$l_4(\mu)l_2(\mu) - l_3^2(\mu) = 0 \tag{2.13}$$

with

$$\begin{aligned} l_2(\mu) &= \beta_{55}\mu^2 - 2\beta_{45}\mu + \beta_{44} \\ l_3(\mu)&\beta_{15}\mu^3 - (\beta_{14}+\beta_{56})\mu^2 + (\beta_{25}+\beta_{46})\mu - \beta_{24} \\ l_4(\mu) &= \beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12}+\beta_{66})\mu^2 - 2\beta_{26}\mu + \beta_{22} \end{aligned} \tag{2.14}$$

It has been shown that[8] the $\mu_n(n = 1, 2, \dots 6)$ which are the roots of (2.13), depend heavily on the elastic constants of the material and are either complex or purely imaginary.

Considering the geometry of the structure, the cylindrical coordinates are the natural coordinates of the problem. Thus, if we can transform all field quantities to the cylindrical coordinates (r, θ, z), then the expression of the boundary conditions will become extremely simple. After some algebra it can be shown that, in the cylindrical coordinate system the governing eqn (2.11), takes the following form:

$$\bar{D}_6\bar{D}_5\bar{D}_4\bar{D}_3\bar{D}_2\bar{D}_1\,\phi^* = 0 \tag{2.15}$$

where the linear operators $\bar{D}_n$, are given by:

$$\bar{D}_n = (\sin\theta - \mu_n\cos\theta)\frac{\partial}{\partial r} + (\cos\theta + \mu_n\sin\theta)\frac{1}{r}\frac{\partial}{\partial\theta} \quad (n = 1, 2, \dots 6). \tag{2.16}$$

and the stress components can be expressed as,

$$\sigma_r = \frac{1}{r}\frac{\partial\phi^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi^*}{\partial\theta^2},\ \sigma_\theta = \frac{\partial^2\phi^*}{\partial r^2}\ \tau_{r\theta} = -\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\phi^*}{r}\right),\ \tau_{rz} = \frac{1}{r}\frac{\partial\psi^*}{\partial\theta}\ \tau_{\theta z} = -\frac{\partial\psi^*}{\partial r} \tag{2.17}$$

where ϕ^* and ψ^* are the new stress functions. Assuming that the μ_n are distinct, the general solution of (2.15) is of the following form:

$$\phi^* = \sum_{n=1}^{6} F_n^*[r(\cos\theta + \mu_n \sin\theta)] \tag{2.18}$$

$$\psi^* = -\sum_{n=1}^{6} \frac{l_3(\mu_n)}{l_2(\mu_n)} \frac{\partial \phi^*}{\partial z_n}[r(\cos\theta + \mu_n \sin\theta)] \tag{2.19}$$

where z_n denotes the argument $r(\cos\theta + \mu_n \sin\theta)$. If one assumes that the functions F_n^* ($n = 1, 2, \ldots 6$) can be written as power series in terms of the argument $r(\cos\theta + \mu_n \sin\theta)$, and considering the fact that we are looking for a singular type solution, it is then sufficient to consider the leading term of the series only. Thus, assuming that the power of the leading term in the series is λ, the stress functions ϕ^* and ψ^* may be expressed as:

$$\phi^* = \sum_{n=1}^{6} A_n r^\lambda (\cos\theta + \mu_n \sin\theta)^\lambda \tag{2.20}$$

$$\psi^* = \sum_{n=1}^{6} \lambda \delta_n A_n r^{\lambda-1} (\cos\theta + \mu_n \sin\theta)^{\lambda-1} \tag{2.21}$$

where A_n ($n = 1, \ldots. 6$) are unknown constants and,

$$\delta_n = -\frac{l_3(\mu_n)}{l_2(\mu_n)}, \quad n = 1, 2, \ldots 6.$$

Using expressions (2.17), the stresses are obtained as:

$$\begin{aligned}
\sigma_r &= \sum_{n=1}^{6} \lambda(\lambda-1) A_n r^{\lambda-2}(-\sin\theta + \mu_n \cos\theta)^2 (\cos\theta + \mu_n \sin\theta)^{\lambda-2} \\
\sigma_\theta &= \sum_{n=1}^{6} \lambda(\lambda-1) A_n r^{\lambda-2} (\cos\theta + \mu_n \sin\theta)^\lambda \\
\tau_{r\theta} &= -\sum_{n=1}^{6} \lambda(\lambda-1) A_n r^{\lambda-2}(-\sin\theta + \mu_n \cos\theta)(\cos\theta + \mu_n \sin\theta)^{\lambda-1} \\
\tau_{\theta z} &= -\sum_{n=1}^{6} \lambda(\lambda-1) A_n \delta_n r^{\lambda-2} (\cos\theta + \mu_n \sin\theta)^{\lambda-1} \\
\tau_{rz} &= \sum_{n=1}^{6} \lambda(\lambda-1) A_n \delta_n r^{\lambda-2}(-\sin\theta + \mu_n \cos\theta)(\cos\theta + \mu_n \sin\theta)^{\lambda-1}.
\end{aligned} \tag{2.22}$$

The other field quantities needed in the formulation of the problem are the components of the displacement vector. The cartesian components of the displacement vector u, v, w are given by[8]:

$$\begin{aligned}
u &= \sum_{n=1}^{6} \lambda A_n p_n r^{\lambda-1} (\cos\theta + \mu_n \sin\theta)^{\lambda-1} \\
v &= \sum_{n=1}^{6} \lambda A_n q_n r^{\lambda-1} (\cos\theta + \mu_n \sin\theta)^{\lambda-1} \\
w &= \sum_{n=1}^{6} \lambda A_n s_n r^{\lambda-1} (\cos\theta + \mu_n \sin\theta)^{\lambda-1}
\end{aligned} \tag{2.23}$$

where the expressions (2.20) and (2.21) are used in deriving eqns (2.23) and the constants p_n, q_n, and s_n are expressed as:

$$p_n = \beta_{11}\mu_n^2 + \beta_{12} - \beta_{16}\mu_n + \delta_n(\beta_{15}\mu_n - \beta_{14})$$

$$q_n = \beta_{12}\mu_n + \frac{\beta_{22}}{\mu_n} + \beta_{26} + \delta_n\left(\beta_{25} - \frac{\beta_{24}}{\mu_n}\right)$$

$$s_n = \beta_{14}\mu_n + \frac{\beta_{24}}{\mu_n} - \beta_{46} + \delta_n\left(\beta_{45} - \frac{\beta_{44}}{\mu_n}\right)$$

$$(n = 1, 2, \ldots 6).$$

The components of the displacement vector in cylindrical coordinates (u_r, u_θ, u_z) can be obtained by the following simple transformation:

$$\begin{aligned} u_r &= u\cos\theta + v\sin\theta \\ u_\theta &= -u\sin\theta + v\cos\theta \\ u_z &= w. \end{aligned} \tag{2.25}$$

Thus,

$$\begin{aligned} u_r &= \sum_{n=1}^{6} \lambda A_n C_{rn} r^{\lambda-1}\left(\cos\theta + \mu_n \sin\theta\right)^{\lambda-1} \\ u_\theta &= \sum_{n=1}^{6} \lambda A_n C_{\theta n} r^{\lambda-1}\left(\cos\theta + \mu_n \sin\theta\right)^{\lambda-1} \\ u_z &= \sum_{n=1}^{6} \lambda A_n C_{zn} r^{\lambda-1}\left(\cos\theta + \mu_n \sin\theta\right)^{\lambda-1} \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} C_{rn} &= p_n\cos\theta + q_n\sin\theta \\ C_{\theta n} &= -p_n\sin\theta + q_n\cos\theta \\ C_{zn} &= s_n \end{aligned} \tag{2.27}$$

$$(n = 1, 2, \ldots 6)$$

3. APPLICATION TO THE TWO-WEDGE PROBLEM

Consider the two bonded anisotropic wedges shown in Fig. 1. The bounding surfaces are stress-free and it is further assumed that the structure is loaded at infinity with self-equilibrating forces. Thus, the following homogeneous boundary and continuity conditions can be written:

$$\sigma_\theta^a(r,0) = \sigma_\theta^b(r,0) \tag{3.1}$$

$$\tau_{r\theta}^a(r,0) = \tau_{r\theta}^b(r,0) \tag{3.2}$$

$$\tau_{\theta z}^a(r,0) = \tau_{\theta z}^b(r,0) \tag{3.3}$$

$$u_r^a(r,0) = u_r^b(r,0) \tag{3.4}$$

$$u_\theta^a(r,0) = u_\theta^b(r,0) \tag{3.5}$$

$$u_z^a(r,0) = u_z^b(r,0) \tag{3.6}$$

$$\sigma_\theta^b(r,\alpha) = 0 \tag{3.7}$$

$$\tau_{r\theta}^{b}(r, \alpha) = 0 \tag{3.8}$$

$$\tau_{\theta z}^{b}(r, \alpha) = 0 \tag{3.9}$$

$$\sigma_{\theta}^{a}(r, -\phi) = 0 \tag{3.10}$$

$$\tau_{r\theta}^{a}(r, -\phi) = 0 \tag{3.11}$$

$$\tau_{\theta z}^{a}(r, -\phi) = 0. \tag{3.12}$$

Using expressions (2.22) for the stresses and (2.26) for the displacements, the boundary and continuity conditions (3.1)–(3.12) give:

$$\sum_{n=1}^{6} A_n^{a} - \sum_{n=1}^{6} A_n^{b} = 0 \tag{3.13}$$

$$\sum_{n=1}^{6} A_n^{a} \mu_n^{a} - \sum_{n=1}^{6} A_n^{b} \mu_n^{b} = 0 \tag{3.14}$$

$$\sum_{n=1}^{6} A_n^{a} \delta_n^{a} - \sum_{n=1}^{6} A_n^{b} \delta_n^{b} = 0 \tag{3.15}$$

$$\sum_{n=1}^{6} A_n^{a} p_n^{a} - \sum_{n=1}^{6} A_n^{b} p_n^{b} = 0 \tag{3.16}$$

$$\sum_{n=1}^{6} A_n^{a} q_n^{a} - \sum_{n=1}^{6} A_n^{b} q_n^{b} = 0 \tag{3.17}$$

$$\sum_{n=1}^{6} A_n^{a} s_n^{a} - \sum_{n=1}^{6} A_n^{b} s_n^{b} = 0 \tag{3.18}$$

$$\sum_{n=1}^{6} A_n^{b} \left(\cos\alpha + \mu_n^{b} \sin\alpha\right)^{\lambda} = 0 \tag{3.19}$$

$$\sum_{n=1}^{6} A_n^{b} (-\sin\alpha + \mu_n^{b} \cos\alpha)\left(\cos\alpha + \mu_n^{b} \sin\alpha\right)^{\lambda-1} = 0 \tag{3.20}$$

$$\sum_{n=1}^{6} A_n^{b} \delta_n^{b} \left(\cos\alpha + \mu_n^{b} \sin\alpha\right)^{\lambda-1} = 0 \tag{3.21}$$

$$\sum_{n=1}^{6} A_n^{a} \left(\cos\phi - \mu_n^{a} \sin\phi\right)^{\lambda} = 0 \tag{3.22}$$

$$\sum_{n=1}^{6} A_n^{a} (\sin\phi + \mu_n^{a} \cos\phi)(\cos\phi - \mu_n^{a} \sin\phi)^{\lambda-1} = 0 \tag{3.23}$$

$$\sum_{n=1}^{6} A_n^{a} \delta_n^{a} \left(\cos\phi - \mu_n^{a} \sin\phi\right)^{\lambda-1} = 0 \tag{3.24}$$

where A_n^a and A_n^b ($n = 1, 2, \ldots 6$) are unknown constants. Equations (3.13)–(3.24) constitute a 12×12 system of homogeneous equations for the 12 unknowns A_n^a and A_n^b. For a non-trivial solution the determinant of coefficients must vanish, thus giving the following characteristic equation:

$$\Delta(\lambda) = 0. \tag{3.25}$$

Equation (3.25) is a transcendental equation in λ and has infinite number of roots which can be determined numerically.

4. SOLUTION AND RESULTS

As stated earlier, the main objective of this study is to determine the power of singularity near the free edge of two materials. Equation (3.25) has infinite number of roots. By imposing the physical condition that the strain energy be bounded in a finite region around the bi-material interface, it can easily be shown that the real part of λ must be greater than one. Furthermore, a singular stress-state will prevail if only $\mathrm{Re}(\lambda) < 2$. Therefore, it is sufficient to search for a root whose real part is the smallest and satisfies the following condition:

$$1 < \mathrm{Re}(\lambda) < 2. \tag{4.1}$$

We assume that each layer is a fiber reinforced composite, such that the fibers lie in the y–z plane, and make the angles $-\theta_\alpha^*$ with the y axis. (The angles θ_α^* ($\alpha = a, b$) are measured from the y-axis in the counter-clockwise direction.) The engineering constants of the fiber reinforced composite are given as follows:

$$\begin{aligned} E_L &= 163.4 \times 10^9\,\mathrm{N/m^2} & \nu_{LT} &= 0.3 \\ E_T &= 11.9 \times 10^9\,\mathrm{N/m^2} & \nu_{LZ} &= 0.3 \\ E_z &= 11.9 \times 10^9\,\mathrm{N/m^2} & \nu_{TZ} &= 0.5 \\ G_{LT} &= 6.5 \times 10^9\,\mathrm{N/m^2} & & \\ G_{ZL} &= 6.5 \times 10^9\,\mathrm{N/m^2} & & \\ G_{ZT} &= 3.5 \times 10^9\,\mathrm{N/m^2} & & \end{aligned}$$

The constants a_{ij} ($i = 1, \ldots 6$, $j = 1, \ldots 6$) which appear in eqns (2.1) can be obtained by rotating the material by an angle θ_α^* about the x-axis. Thus, the composite material which can be considered as orthotropic in the (L, T, Z) coordinate system is fully anisotropic in the (x, y, z) coordinate system. Since the solution given in this paper breaks down for orthotropic materials,† the following restrictions are imposed on the angles θ_α^*:

$$\theta_\alpha^* \neq 0^\circ, \quad \theta_\alpha^* \neq \pm 90^\circ \text{ and } \theta_a^* \neq \theta_b^* \tag{4.2}$$

Using the general two-wedge formulation, the following problems are studied:

(a) Two bonded layers with a stress free-edge (Fig. 1a): This problem arises in studying the edge effects or boundary layer effects in composites. For this case, the wedge angles are taken as $\alpha = 90^\circ$ and $\phi = 90^\circ$. The stress singularity is computed for different composite materials obtained by varying the angle θ_b^* that the fibers make with the y-axis (Table 1). The results show that the power of singularity ($\lambda - 2$) is real, and although it can vary from material to material, remains relatively small.

(b) The case of a broken laminate (Fig. 2b): This geometry is obtained by letting $\alpha = 90^\circ$ and $\phi = 180^\circ$. The variation of the power of singularity is given in Table 2. It can be observed that for this case the power of singularity ($\lambda - 2$) is again real, close to -0.5 and is much higher than that obtained for the previous case. The implication of this is that, from the viewpoint of delamination failure the case of a broken laminate appears to be more severe than that of a free-edge.

(c) A wedge bonded to a half-plane (Fig. 2c): In this case the ply angles θ_a^* and θ_b^* are fixed and the effect of the wedge angle α is investigated. The results are given in Table 3 and are displayed in Fig. 3. As it can be observed from Fig. 3, the power of singularity becomes more severe as the wedge angle increases, reaching the value of $-1/2$ for a crack between two anisotropic materials. For the particular material used in this paper, the power of singularity is real for all values of the wedge angle α. However, for isotropic materials, it is known that the power of singularity at the crack tip is complex (i.e. of oscillatory nature) with real part equal to $-1/2$ (see for example [12]). A complex power of singularity has for many years puzzled researchers about its physical and mathematical meaning. Even though the physical explanation of the complex power of singularity may still remain obscure, the results presented in this

†For orthotropic materials differential eqns (2.5) and (2.6) uncouple, meaning that the plane elasticity problem and the anti-plane problem must be formulated independently. Thus the roots μ_n given by the characteristic eqn (2.13) cannot be used to construct the general solution.

Table 1. Variation of the power of singularity (λ-2) near the free edge of two layers with θ_b* (Fig. 2a, with θ_a* = 30°)

θ_b^*	-75°	-60°	-45°	-30°	-15°	15°	45°	60°	75°
λ - 2	-0.0731	-0.0582	-0.0388	-0.0256	-0.0235	-0.0113	-0.0167	-0.0507	-0.0733

Table 2. Variation of the power of singularity (λ-2) near the edge of a broken laminate with θ_b* (Fig. 2b, with θ_a* = 30°)

θ_b^*	-75°	-60°	-45°	-30°	-15°	15°	45°	60°	75°
λ - 2	-0.4184	-0.4333	-0.4490	-0.4578	-0.4614	-0.4559	-0.4355	-0.4229	-0.4187

Table 3. Variation of the power of singularity (λ – 2) near the vertex of a wedge bonded to a half-plane with the wedge angle. (Fig. 2c, θ_a* = 30°, θ_b* = 60°)

α	5°	10	20	30	60	90°	120°	150	160	170	175	178	179°
(λ-2)	-0.0408	-0.0785	-0.1465	-0.2050	-0.3356	-0.4229	-0.4743	-0.4888	-0.4900	-0.4947	-0.4974	-0.4990	-0.4995

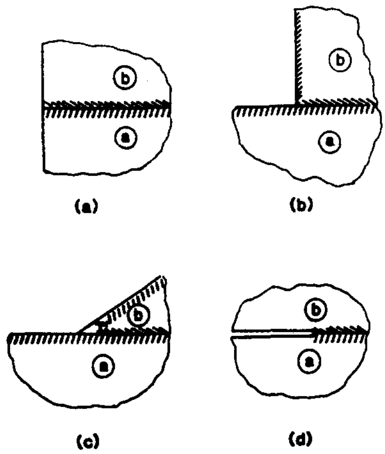

Fig. 2. Geometry of problems considered in the study.

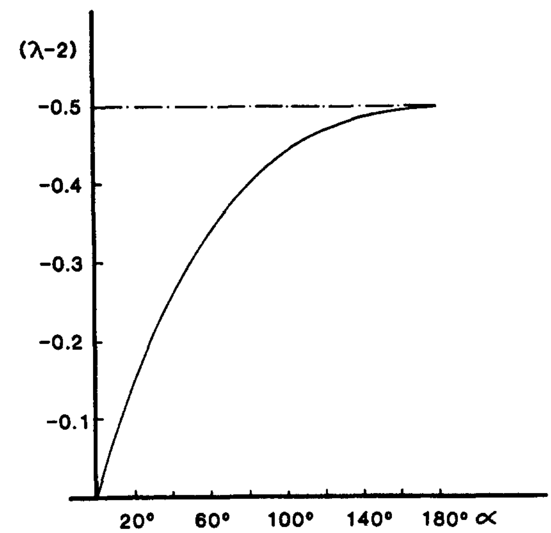

Fig. 3. Variation of the power of singularity (λ-2) with the wedge angle α, for a wedge bonded to a half-plane ($\theta_a^* = 30°$, $\theta_b^* = 60°$).

paper, together with the explanation given in[4,5] may shed some light on the mathematical explanation of the phenomenon†. The characteristic equation (3.25) has infinite number of roots some real, some complex. These roots depend heavily on the elastic constants of the anisotropic constituents and the wedge angle. As shown in [4, 5] for orthotropic materials the dominant root may be real or complex. Therefore, the oscillatory singularity found for a crack at a bimaterial interface is by no means a universal fact. For anisotropic materials the power of singularity, as shown in this paper, may very well be real.

(d) A crack at the interface of two anisotropic materials (Fig. 2d): This configuration is obtained as a limiting case of the problem studied in (c) by letting $\alpha \rightarrow 180°$. Since for $\alpha = 180°$ the system of equation becomes unstable, the power of singularity in this case is obtained by a limiting process, carried out by assigning to α closer and closer values of 180°. As stated earlier, for this configuration the power of singularity is found to be real and equal to $-1/2$. The results obtained in this paper are of great importance in performing the stress analysis of layered composite structures. In real structures, generally the geometry is very complex to lend itself to

†It may be worthwhile to note that the oscillatory singularity found for isotropic materials is a direct consequence of the linear theory used in modeling the material. It has been shown that, this anomaly disappears when the non-linear theory of harmonic elastic materials is used[14].

analytical formulation. Therefore, one has to resort to purely numerical techniques, such as the finite element method. As shown in this paper, near free-edges, at the base of a broken laminate, or near a crack tip between two layers the stress state is singular and the results obtained by ignoring the singularity can be very erroneous, specially when computing physical quantities near the singular field. Therefore, near a singular stress field, special elements must be designed and incorporated into the finite element grid. The asymptotic expressions given in this paper for the displacements and the strains can be used to formulate those special elements.

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